

Generalized Fano and non-Fano networks

Niladri Das and Brijesh Kumar Rai

Department of Electronics and Electrical Engineering
Indian Institute of Technology Guwahati, Guwahati, Assam, India
Email: {d.niladri, bkrai}@iitg.ernet.in

Abstract—It is known that the Fano network has a vector linear solution if and only if the characteristic of the finite field is 2; and the non-Fano network has a vector linear solution if and only if the characteristic of the finite field is not 2. Using these properties of Fano and non-Fano networks it has been shown that linear network coding is insufficient. In this paper we generalize the properties of Fano and non-Fano networks. Specifically, by adding more nodes and edges to the Fano network, we construct a network which has a vector linear solution for any vector dimension if and only if the characteristic of the finite field belongs to an arbitrary given set of primes $\{p_1, p_2, \dots, p_l\}$. Similarly, by adding more nodes and edges to the non-Fano network, we construct a network which has a vector linear solution for any vector dimension if and only if the characteristic of the finite field does not belong to an arbitrary given set of primes $\{p_1, p_2, \dots, p_l\}$.

I. INTRODUCTION

Network coding refers to a data transmission scheme whereby instead of treating data symbols as a commodity, the intermediate nodes forward data which are functions of incoming data symbols. By using such a transmission scheme it has been shown that the min-cut upper bound on the capacity of multicast networks can be achieved [1]. Linear network coding refers to the network coding scheme where all the nodes are restricted to compute only linear functions. More precisely, in linear network coding, the data symbols generated by the sources are assumed to belong to a finite field say \mathbb{F}_q , and nodes transmit a \mathbb{F}_q -linear combination of the data symbols it receives.

It is known that the linear coding capacity of a network can be dependent on the characteristic of the finite field [2]. In [2] Dougherty *et al.* constructed a network, named as the Fano network, which has a scalar linear solution over any finite field of characteristic 2, but has no vector linear solution for any vector dimension over any finite field of odd characteristic. By adding more nodes and edges to the same network we show that for any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, the resulting network has (a scalar linear solution if the characteristic belong to the given set of primes) a vector linear solution for any vector dimension if and only if the characteristic of the finite field belong to the given set.

Along with the Fano network, Dougherty *et al.* also presented the non-Fano network in [3] which has a scalar linear solution if the characteristic of the finite field is any prime number other than 2, but has not vector linear solution for any vector dimension if the characteristic of the finite field is 2. We use the same network as a sub-network to construct a network

which for any given set of prime numbers $\{p_1, p_2, \dots, p_l\}$ has a vector linear solution if and only if the characteristic of the finite field does not belong to the given set.

Both of the Fano and the non-Fano networks have been constructed from the Fano and non-Fano matroid respectively. Considering the equivalence between matroids and linear network coding presented in [3] and [4], our results also show that from the Fano and non-Fano matroid itself, matroids can be constructed which are representable if and only if the characteristic of the finite field belong to some arbitrary finite or co-finite set of primes. A combination of the Fano and non-Fano network has been used in [2] to show that linear network coding is insufficient. The achievable rate region for the Fano and non-Fano networks were described in reference [5].

It is to be noted that in references [6] and [7] it has been already shown that for any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, there exists a network which has a vector linear solution for any vector dimension if and only if the characteristic of the finite field belong to the given set. The authors first proved the results for sum-networks, and then showed that for every sum-network there exists a linear solvably equivalent multiple-unicast network. Similarly, for any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, they constructed a sum-network which has a vector linear solution if and only if the characteristic of the finite field does not belong to the given set. Invoking the same linear equivalence between multiple-unicast networks and sum-networks they showed the same properties hold for multiple-unicast networks. However, to the best of our knowledge this is the first time it is shown that the Fano-network and the non-Fano network itself can be used as a sub-network to construct a network having the same properties with an added desirability that these networks are simpler as they have less number of sources, terminals, nodes and edges.

The organization of the paper is as follows. In Section II we reproduce the standard definition of vector linear network coding. In Section III, we present the generalized Fano network, and in the following section, Section IV, we present the generalized non-Fano network. We conclude the paper in Section V.

II. PRELIMINARIES

We represent a network by a graph $G(V, E)$. Some of the nodes are considered as sources and some as terminals. It is assumed that the sources have no incoming edges and the terminals have no outgoing edges. The set of sources is denoted by S ; and the set of terminals is denoted by T . Every

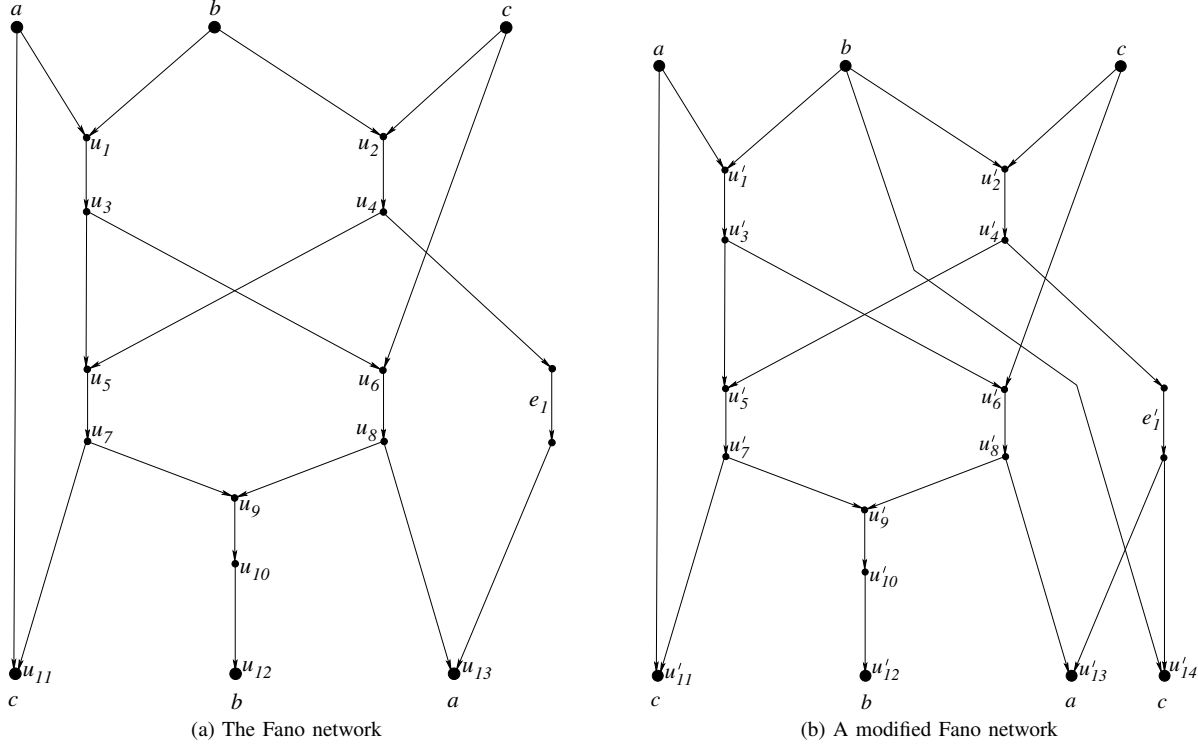


Fig. 1. A modified Fano network which has an l -dimensional vector linear solution if and only if the Fano network has an l -dimensional vector linear solution.

source generates an i.i.d. random process uniformly distributed over a finite alphabet. In linear network coding this alphabet is assumed to be a finite field \mathbb{F}_q . Any source process is independent of all other source processes. The source process generated by source $s \in S$ is denoted by X_s . Each terminal is required to retrieve the random process generated at some source. The symbol carried by an edge e is denoted by Y_e . A node $v \in V$ is called an intermediate node if it is neither a source node nor a terminal node. $In(v)$ denotes the set of edges e such that $head(e) = v$. $Out(v)$ denotes the set of edges e such that $tail(e) = v$. The i^{th} edge between nodes v_1 and v_2 is denoted by (v_1, v_2, i) . In case there is only one edge between v_1 and v_2 , we denote it by (v_1, v_2) .

A k -dimensional vector linear network coding is defined as follows. If $e \in Out(s)$ for any $s \in S$, then $Y_e = A_{s,e} X_s$ where $Y_e, X_s \in \mathbb{F}_q^k$ and $A_{s,e} \in \mathbb{F}_q^{k \times k}$. For any intermediate node v , if $e \in Out(v)$ and $In(v) = \{e_1, e_2, \dots, e_n\}$, then $Y_e = A_{e_1,v} Y_{e_1} + A_{e_2,v} Y_{e_2} + \dots + A_{e_n,v} Y_{e_n}$, where $Y_e, Y_{e_1}, Y_{e_2}, \dots, Y_{e_n} \in \mathbb{F}_q^k$ and $A_{e_i,v} \in \mathbb{F}_q^{k \times k}$ for $1 \leq i \leq n$. Each terminal $t \in T$ can compute a block of k symbols from a source as $Y_t = A_{e_1,t} Y_{e_1} + A_{e_2,t} Y_{e_2} + \dots + A_{e_n,t} Y_{e_n}$ where $In(t) = \{e_1, e_2, \dots, e_n\}$, $Y_{e_1}, Y_{e_2}, \dots, Y_{e_n} \in \mathbb{F}_q^k$ and $A_{e_i,t} \in \mathbb{F}_q^{k \times k}$ for $1 \leq i \leq n$.

If all terminals, by obeying the above restrictions, can compute a block of k symbols from their respective sources in k usages of the network, then the network is said to have a k -dimensional vector linear solution, or analogously, the network is said to be vector linearly solvable for k vector dimensions. In a k -dimensional vector linear solution, k is called as the

vector dimension. When the network has an 1-dimensional vector linear solution the network is said to be scalar linearly solvable. The matrices indicated above by $A_{a,b}$ where a and b is either a node or an edge are called as local coding matrices.

III. GENERALIZED FANO NETWORK

In this section, for any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, a network is constructed, by adding more nodes and edges to the Fano network, which has a vector linear solution for any message dimension if and only if the characteristic of the finite field belong to the set $\{p_1, p_2, \dots, p_l\}$. The Fano network shown in [2] was constructed using an algorithm given in [3] that takes the Fano matroid as an input. Considering that there is no unique algorithm to construct a network from a matroid, and also that the construction algorithm given in [3] itself leaves the scope of adding more terminals, we have found that a modified Fano network (which will be shown to be linear solvably equivalent to the Fano network) to be the one which is the special case of the generalized Fano network constructed in this paper. Towards this end, we present a modified Fano network which has an l -dimensional vector linear solution if and only if the Fano network has an l -dimensional vector linear solution. The Fano network reproduced from [2] is shown in Fig. 1a and the modified Fano network is shown in Fig. 1b. Each of these two networks has three sources which generate the random processes a, b and c respectively. The nodes at the bottom are the terminal nodes. Each terminal demands one of the source processes from a, b and c as

indicated in the figures. To note that the modified Fano network is constructed by adding two edges and one terminal to the Fano network.

Lemma 1. *The network shown in Fig. 1b has an l -dimensional vector linear solution if and only if the Fano network has an l -dimensional vector linear solution.*

Proof: We first show that if the Fano network has an l -dimensional vector linear solution then the modified Fano network also has an l -dimensional vector linear solution. Here we give the proof for the scalar linear solution, i.e., we prove that if the Fano network has a scalar linear solution then the modified Fano network also has a scalar linear solution. Say $Y_{e_1} = \alpha b + \beta c$. It we can show that $\beta \neq 0$, then upon receiving b from the direct edge, node u'_{14} in the modified Fano network can compute c . Say $\beta = 0$. Since c is computed at u_{11} , and (u_2, u_4) disconnects the source that generates c and node u_{11} , the coefficient multiplying c in $Y_{(u_2, u_4)}$ cannot be zero. Hence if $\beta = 0$, it means $Y_{(u_2, u_4)}$ has been multiplied by zero, and $Y_{e_1} = 0$. Then, from (u_6, u_8) solely, a must be retrieved by u_{12} . And hence the coefficient of b and c in $Y_{(u_6, u_8)}$ is zero. So b must be retrieved at node u_{12} from (u_5, u_7) solely, and this indicates that the coefficient of c in $Y_{(u_5, u_7)}$ is zero. Therefore, node u_{11} cannot compute c which is a contradiction. So, $\beta \neq 0$. The proof for l -dimensional vector linear solution can be done in a similar way by taking β as $l \times l$ matrix and showing that matrix β has to be a full rank matrix.

The converse that if the modified Fano network has an l -dimensional vector linear solution then the Fano network also has an l -dimensional vector linear solution is immediate because the Fano network is a sub-network of the modified Fano network. ■

We now present the generalized Fano network in Fig. 2. The top nodes are sources, and the sources generate the random processes $a, b_1, b_2, \dots, b_{p-1}$ and c respectively. The local coding matrices in the network are shown alongside the edges in capital letters. To maintain the cleanliness of the figure, some of the direct edges have been depicted by a edge without a tail node along with the notation of the source process it is connected to. The parameter q which determines the number of nodes and edges in the network can take any integer value greater than or equal to two. Note that the network shown in Fig 2 reduces to the modified Fano network shown in Fig. 1b for $q = 2$.

Lemma 2. *The network shown in Fig. 2 has a vector linear solution for any vector dimension if and only if the characteristic of the finite field divides q .*

Proof: We first show that the network in Fig. 2 has a vector linear solution only if the characteristic of the finite field divides q . Say, the network has a k dimensional vector linear solution.

$$Y_{u_1, u_3} = D_1 a + \sum_{i=1}^{q-1} A_i b_i \quad (1)$$

$$Y_{u_2, u_4} = \sum_{i=1}^{q-1} B_i b_i + D_2 c \quad (2)$$

$$\begin{aligned} Y_{u_5, u_7} &= M_1 e_{13} + M_2 e_{24} \\ &= M_1 D_1 a + \sum_{i=1}^{q-1} (M_1 A_i + M_2 B_i) b_i + M_2 D_2 c \end{aligned} \quad (3)$$

$$Y_{u_6, u_8} = M_3 e_{13} + D_3 c = M_3 D_1 a + \sum_{i=1}^{q-1} M_3 A_i b_i + D_3 c \quad (4)$$

$$\begin{aligned} Y_{e_i} &= W_i e_{24} + \sum_{j=1, j \neq i}^{q-1} U_{ji} b_j \quad \text{for } 1 \leq i \leq q-1 \\ &= W_i B_i b_i + \sum_{j=1, j \neq i}^{q-1} (W_i B_j + U_{ji}) b_j + W_i D_2 c \end{aligned} \quad (5)$$

$$\begin{aligned} Y_{u_9, u_{10}} &= M_4 e_{5,7} + M_5 e_{6,8} = (M_4 M_1 D_1 + M_5 M_3 D_1) a \\ &+ \sum_{i=1}^{q-1} (M_4 (M_1 A_i + M_2 B_i) + M_5 M_3 A_i) b_i \\ &+ (M_4 M_2 D_2 + M_5 D_3) c \end{aligned} \quad (6)$$

Since t_1 computes c , for $1 \leq i \leq q-1$, we have:

$$Q_1 M_1 D_1 + D_4 = 0 \quad (7)$$

$$Q_1 (M_1 A_i + M_2 B_i) = 0 \quad (8)$$

$$Q_1 M_2 D_2 = I \quad (9)$$

Since t_i for $2 \leq i \leq q$ computes b_{i-1} , for $1 \leq i, j \leq q-1$ and $j \neq i$, we have:

$$K_i (M_4 M_1 D_1 + M_5 M_3 D_1) = 0 \quad (10)$$

$$K_i (M_4 (M_1 A_i + M_2 B_i) + M_5 M_3 A_i) = I \quad (11)$$

$$K_i (M_4 (M_1 A_j + M_2 B_j) + M_5 M_3 A_j) + J_{ji} = 0 \quad (12)$$

$$K_i (M_4 M_2 D_2 + M_5 D_3) = 0 \quad (13)$$

Since t_{q+1} retrieves a , for $1 \leq i \leq q-1$, we have:

$$Q_2 M_3 D_1 = I \quad (14)$$

$$Q_2 M_3 A_i + R_i W_i B_i + \sum_{j=1, j \neq i}^{q-1} R_j (W_j B_i + U_{ij}) = 0 \quad (15)$$

$$Q_2 D_3 + \sum_{i=1}^{q-1} R_i W_i D_2 = 0 \quad (16)$$

Since t_{q+1+i} for $1 \leq i \leq q-1$ demands c , for $1 \leq i, j \leq q-1$ and $j \neq i$, we have:

$$V_i W_i B_i + E_i = 0 \quad (17)$$

$$V_i (W_i B_j + U_{ji}) = 0 \quad (18)$$

$$V_i W_i D_2 = I \quad (19)$$

From equation (9), we know that Q_1 is invertible. Hence from equation (8), we have for $1 \leq i \leq q-1$:

$$M_1 A_i + M_2 B_i = 0 \quad (20)$$

From equation (11), we know that K_i is invertible for $1 \leq i \leq q-1$. Hence from (10), we have,

$$M_4 M_1 D_1 + M_5 M_3 D_1 = 0 \quad (21)$$

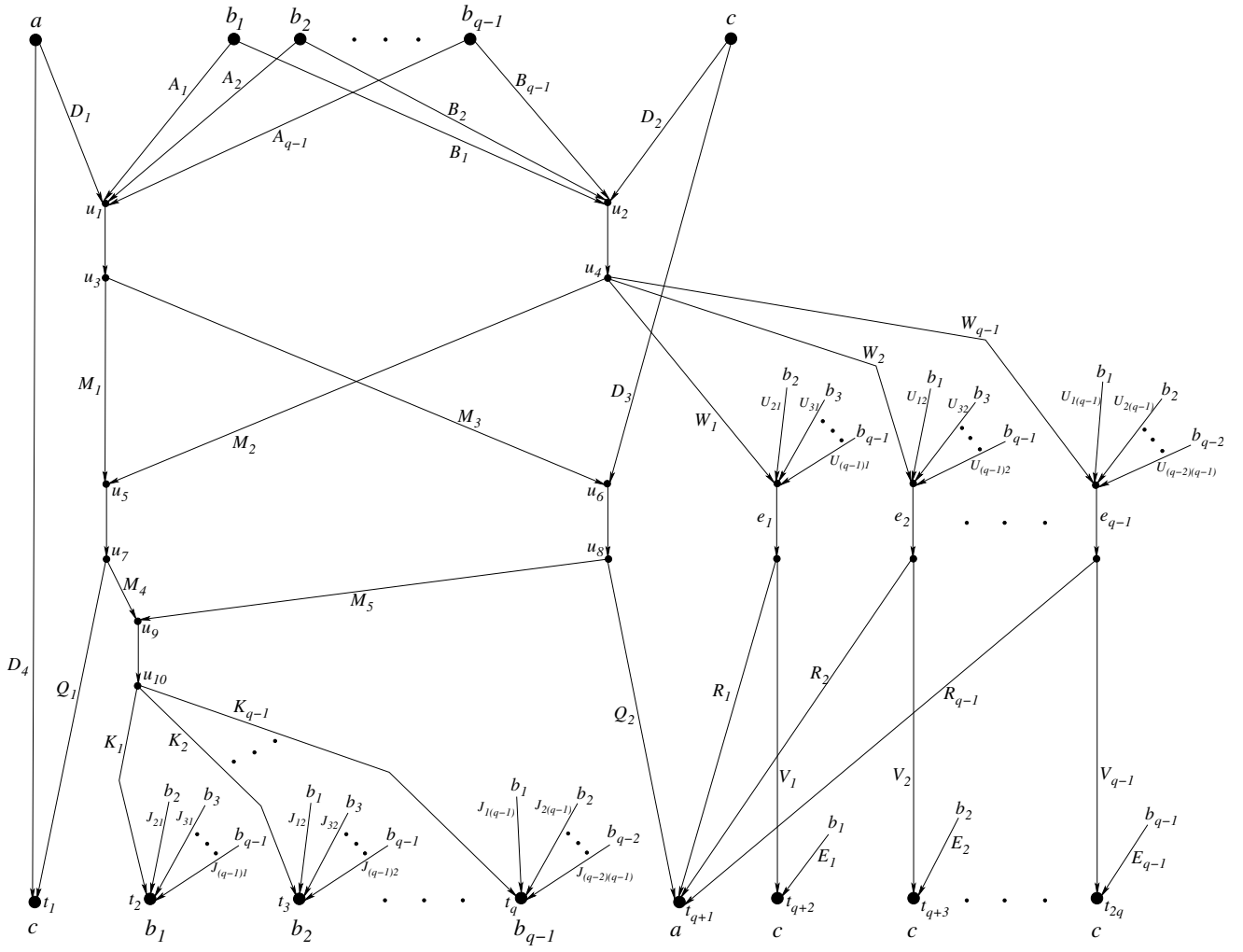


Fig. 2. Generalized Fano network: for any integer $q \geq 2$, the network is vector linearly solvable for any vector dimension if and only if the characteristic of the finite field divides q

It can be seen from equation (14) that D_1 is also invertible. Hence from equation (21),

$$M_4 M_1 + M_5 M_3 = 0 \quad (22)$$

Substituting equation (20) in equation (11), we have for $1 \leq i \leq q-1$:

$$K_i M_5 M_3 A_i = I \quad (23)$$

Also from equation (13), we have:

$$M_4 M_2 D_2 + M_5 D_3 = 0 \quad (24)$$

From equation (19), it can be seen that V_i is invertible for $1 \leq i \leq q-1$. Hence, from equation (18) for $1 \leq i, j \leq q-1$ and $j \neq i$,

$$W_i B_j + U_{ji} = 0 \quad (25)$$

Substituting equation (25) in equation (15) for $1 \leq i \leq q-1$:

$$Q_2 M_3 A_i + R_i W_i B_i = 0 \quad (26)$$

Since D_2 is invertible because of equation (9), from equation (16) we have:

$$Q_2 D_3 D_2^{-1} + \sum_{i=1}^{q-1} R_i W_i = 0 \quad (27)$$

Note that since from equation (23), $M_3 A_i$ is invertible and from equation (14), Q_2 is invertible, $R_i W_i B_i$ in equation (26) invertible, and hence B_i and A_i must be invertible for $1 \leq i \leq q-1$. Also note that M_2 in equation (9) is invertible. Hence, from equation (20), M_1 is invertible. Also from equation (23) M_5 is invertible. Substituting $R_i W_i$ from equation (26) in equation (27) we have:

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 M_3 A_i B_i^{-1} = 0 \quad (28)$$

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 M_3 A_i B_i^{-1} = 0 \quad (29)$$

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 M_3 (-M_1^{-1} M_2) = 0 \quad [\text{from (20)}] \quad (30)$$

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 (-M_3 M_1^{-1}) M_2 = 0 \quad (31)$$

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 M_5^{-1} M_4 M_2 = 0 \quad [\text{from (22)}] \quad (32)$$

$$Q_2 D_3 D_2^{-1} - \sum_{i=1}^{q-1} Q_2 (-D_3 D_2^{-1}) = 0 \quad [\text{from (24)}] \quad (33)$$

$$Q_2 D_3 D_2^{-1} + \sum_{i=1}^{q-1} Q_2 D_3 D_2^{-1} = 0 \quad (34)$$

$$(q) Q_2 D_3 D_2^{-1} = 0 \quad (35)$$

Since Q_2, D_3 and D_2 are all full rank matrices, it must be that $q = 0$. Now note that over any finite field of certain characteristic, q is zero if and only if the characteristic divides q .

We now show that the network has a scalar linear solution over a characteristic which divides q . Consider the following messages to be carried by the edges.

$$\begin{aligned} Y_{u_1, u_3} &= a + \sum_{i=1}^{p-1} b_i \\ Y_{u_2, u_4} &= \sum_{i=1}^{q-1} b_i + c \\ Y_{u_5, u_7} &= Y_{u_1, u_3} - Y_{u_2, u_4} = a - c \\ Y_{u_6, u_8} &= Y_{u_1, u_3} - c = a + \sum_{i=1}^{q-1} b_i - c \\ \text{for } 1 \leq i \leq q-1 : \quad Y_{e_i} &= b_i + c \\ Y_{u_9, u_{10}} &= Y_{u_6, u_8} - Y_{u_5, u_7} = \sum_{i=1}^{q-1} b_i \end{aligned}$$

Now, we show that the terminals can decode their desired random variables as follows. At terminal t_1 , with the operation $a - Y_{u_5, u_7}$, random variable c can be determined. For $1 \leq i \leq q-1$, the terminal t_{1+i} decodes b_i as $\sum_{m=1}^{m-1} b_m - \sum_{j=1, j \neq m}^{p-1} b_j = b_i$. Since all operations are over the finite field of a characteristic which divides q , terminal t_{q+1} decodes a as $Y_{u_6, u_8} - \sum_{i=1}^{p-1} Y_{e_i} = a - qc = a$. For $1 \leq i \leq q-1$, the terminal t_{q+1+i} performs the operation $Y_{e_i} - b_i$ to derive c . ■

Theorem 3. *For any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, there exists a network, constructed by adding more nodes and edges to the Fano network, which has a vector linear solution if and only if the characteristic of the finite field belongs to the given set.*

Proof: The network in Fig. 2 constructed for $q = p_1^{r_1} \cdot p_2^{r_2} \cdot \dots \cdot p_l^{r_l}$, where $r_1, r_2, \dots, r_l \in \mathbb{Z}^+$ is such a network. ■

IV. GENERALIZED NON-FANO NETWORK

In this section, for any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, we add more nodes and edges to the non-Fano network, to construct a network which has a vector linear solution for any vector dimension if only if the characteristic of the finite field does not belong to the given set. For this purpose, we first construct a modified non-Fano network, shown in Fig. 3b, which is linear solvably equivalent to the non-Fano network shown in [3] and reproduced here in Fig. 3a. The nodes at the top which have no incoming edges are the sources, and they not labelled to reduce clumsiness. The source process generated by a source node is indicated above the node. For any terminal, the source processes demanded by the terminal are indicated below the terminal.

Lemma 4. *The modified non-Fano network in Fig. 3b has an l -dimensional vector linear solution if and only if the network in Fig. 3a has an l -dimensional vector linear solution.*

Proof: As the terminal t'_4 in the modified non-Fano network has demands which is a subset of what t_4 demands in the non-Fano network, it is self-evident that if the non-Fano network has an l -dimensional vector linear solution then the modified non-Fano network too has an l -dimensional vector linear solution. We now show that if the modified non-Fano network has a scalar linear solution then the non-Fano network too has a scalar linear solution. Assume a scalar linear solution of modified non-Fano network. It can be seen that if the coefficient of b_2 in $Y_{e'_1}$ is non-zero then b_2 can be retrieved from $Y_{e'_1}$ by the node t'_4 in Fig. 3b as it already knows a . If however, the coefficient of b_2 in $Y_{e'_1}$ had been zero, then the coefficient of b_2 in $Y_{e'_a}$ had also to be zero, as $Y_{e'_a}$ cannot be multiplied by zero at node t'_2 since t'_2 needs to use the information in $Y_{e'_a}$ to compute b_1 . However, if the coefficient of b_2 in $Y_{e'_a}$ is zero, then the node t'_3 in Fig. 3b won't be able to compute b_2 . Similar argument can be used to derive that the coefficient of b_1 in $Y_{e'_2}$ is non-zero. And hence the node t'_4 in Fig. 3b can compute all of a, b_1 and b_2 .

The proof for l -dimensional vector linear solution can be done in a similar way. ■

We now present the generalized non-Fano network in Fig. IV. The source nodes are not labelled in the figure for the sake of cleanliness in the diagram. a, b_1, b_2, \dots, b_q are the random processes generated by the sources. Note that for $1 \leq i \leq q$, there exists no path between $\text{tail}(e_i)$ and the source that generates message b_i . Here also, the parameter q can take any integer value greater than or equal to two. It can be verified that the network shown in Fig. 3b reduces to the modified non-Fano network shown in Fig. 3b for $q = 2$.

Lemma 5. *The network shown in Fig. IV has a vector linear solution for any vector dimension if and only if the characteristic of the finite field does not divide q .*

Proof: We first list the messages carried over by the

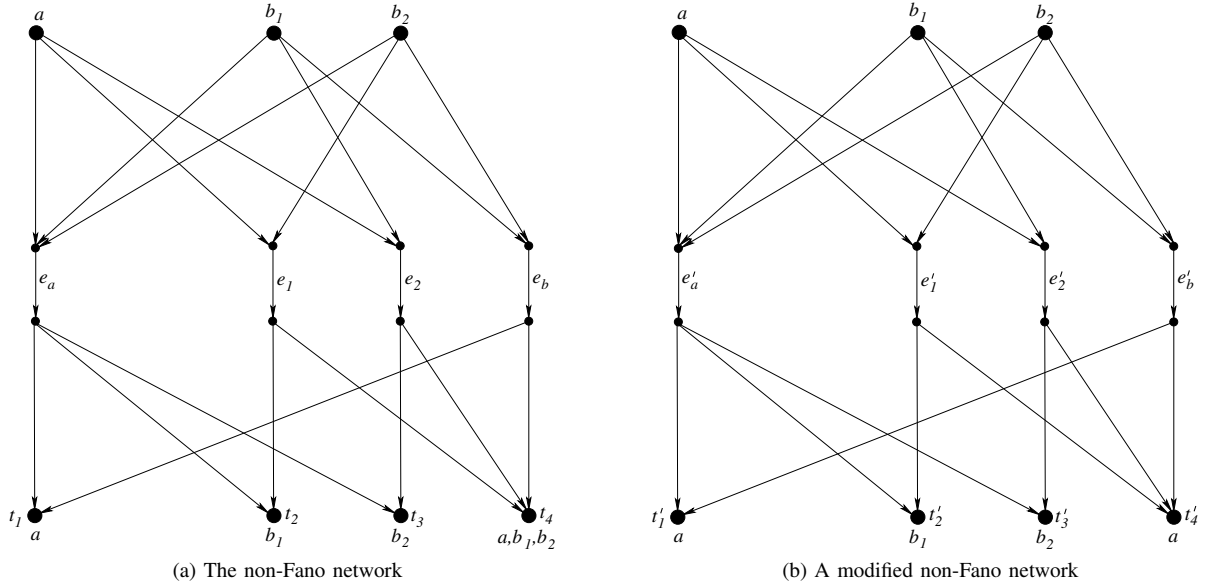


Fig. 3. A modified non-Fano network which has an l -dimensional vector linear solution if and only if the Fano network has an l -dimensional vector linear solution.

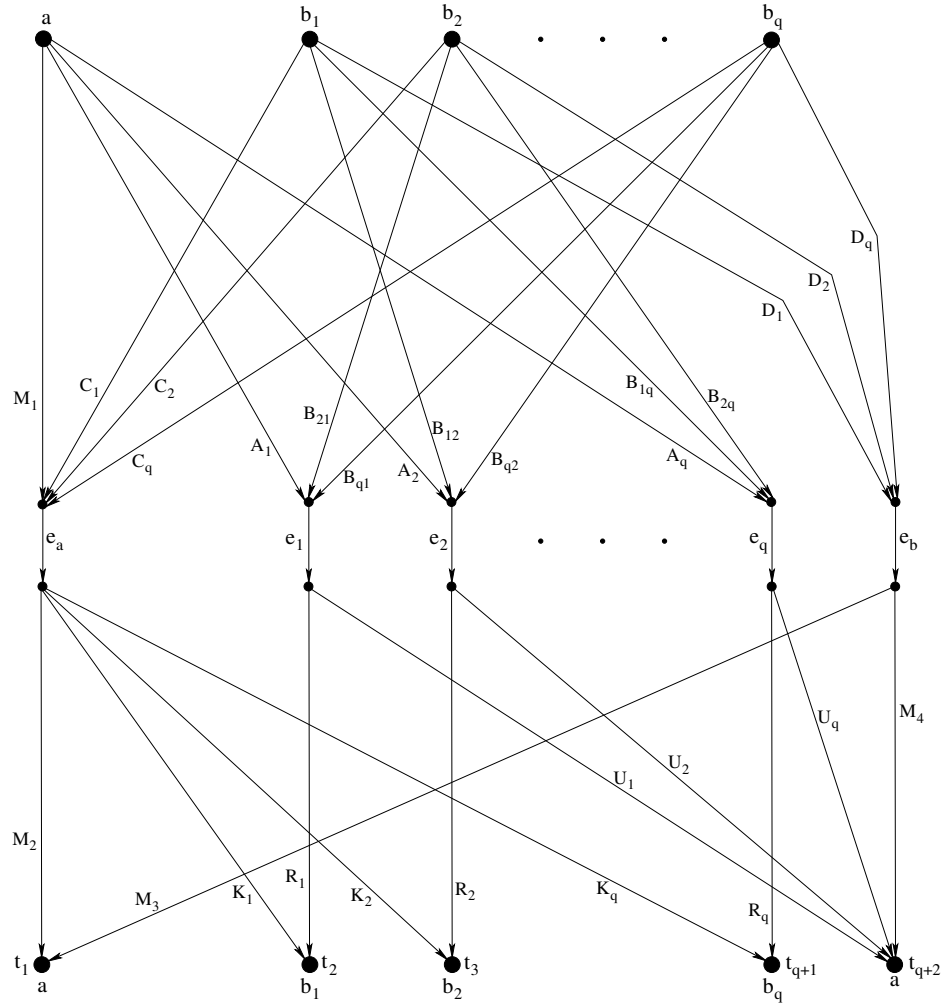


Fig. 4. Generalized non-Fano network: for any integer $q \geq 2$, the network is vector linearly solvable for any vector dimension if and only if the characteristic of the finite field does not divide q .

edges.

$$Y_{e_a} = M_1 a + \sum_{i=1}^q C_i b_i$$

$$\text{for } 1 \leq i \leq q: Y_{e_i} = A_i a + \sum_{j=1, j \neq i}^q B_{ji} b_j$$

$$Y_{e_b} = \sum_{i=1}^q D_i b_i$$

Since, node t_1 computes a , for $1 \leq i \leq q$ we have:

$$M_2 M_1 = I \quad (36)$$

$$M_2 C_i + M_3 D_i = 0 \quad (37)$$

Since, node t_{i+1} for $1 \leq i \leq q$ computes b_i , for $1 \leq i, j \leq q$, $j \neq i$ we have:

$$K_i M_1 + R_i A_i = 0 \quad (38)$$

$$K_i C_i = I \quad (39)$$

$$K_i C_j + R_i B_{ji} = 0 \quad (40)$$

Since, node t_{q+2} computes a , for $1 \leq i \leq q$ we have,

$$\sum_{i=1}^q U_i A_i = I \quad (41)$$

$$\left(\sum_{j=1, j \neq i}^q U_j B_{ij} \right) + M_4 D_i = 0 \quad (42)$$

Since, from equation (39), for $1 \leq i \leq q$, C_i is invertible, and M_2 is invertible from equation (36), $M_3 D_i$ for $1 \leq i \leq q$ is invertible from equation (37), and hence M_3 is invertible. Also, since both of K_i and C_i are invertible for $1 \leq i \leq q$ because of equation (39), R_i and B_{ji} for $1 \leq i, j \leq q$, $j \neq i$ are invertible from equation (40). Moreover, note that M_1 is invertible from equation (36). Now substituting equation (37) in equation (42) we get for $1 \leq i \leq q$:

$$\begin{aligned} & \left(\sum_{j=1, j \neq i}^q U_j B_{ij} \right) - M_4 M_3^{-1} M_2 C_i = 0 \\ & - \left(\sum_{j=1, j \neq i}^q U_j R_j^{-1} K_j C_i \right) - M_4 M_3^{-1} M_2 C_i = 0 \quad [\text{from 40}] \\ & \left(\sum_{j=1, j \neq i}^q U_j A_j M_1^{-1} C_i \right) - M_4 M_3^{-1} M_2 C_i = 0 \quad [\text{from 38}] \\ & \left(\sum_{j=1, j \neq i}^q U_j A_j M_2 C_i \right) - M_4 M_3^{-1} M_2 C_i = 0 \quad [\text{from 36}] \\ & \left(\sum_{j=1, j \neq i}^q U_j A_j - M_4 M_3^{-1} \right) M_2 C_i = 0 \\ & (I - U_i A_i - M_4 M_3^{-1}) M_2 C_i = 0 \quad [\text{from 41}] \\ & I - U_i A_i - M_4 M_3^{-1} = 0 \\ & U_i A_i + M_4 M_3^{-1} = I \\ & U_i A_i = I - M_4 M_3^{-1} \end{aligned} \quad (43)$$

Now, substituting equation (43) in equation (41) we get:

$$\begin{aligned} & \sum_{i=1}^q (I - M_4 M_3^{-1}) = I \\ & qI - qM_4 M_3^{-1} = I \\ & qM_4 M_3^{-1} = (q-1)I \end{aligned} \quad (44)$$

Now, if the characteristic of the finite field divides q , then $q = 0$, and equation (44) results into $0 = -I$, which is a contradiction. We now show that the network in Fig. IV has a scalar linear solution if the characteristic of the finite field is does not divides q . Note that an element in a finite field has an inverse if and only if the characteristics of the finite field does not divide that element. Consider the following messages to be transmitted by the edges:

$$Y_{e_a} = a + \sum_{i=1}^q b_i$$

$$\text{for } 1 \leq i \leq q: Y_{e_i} = a + \sum_{j=1, j \neq i}^q b_j$$

$$Y_{e_b} = \sum_{i=1}^q b_i$$

We now show that the terminals can compute their respective demands. The terminal t_1 computes a as $Y_{e_a} - Y_{e_b} = a$. For $1 \leq i \leq q$, the terminal t_{1+i} decodes b_i by the operation $Y_{e_a} - Y_{e_i}$. At terminal t_{q+2} , since q has an inverse in the finite field, $q^{-1}(\sum_{i=1}^q Y_{e_i} - (q-1)Y_{e_b}) = a$. ■

Theorem 6. For any set of prime numbers $\{p_1, p_2, \dots, p_l\}$, there exists a network, constructed by adding more nodes and edges to the non-Fano network, which has a vector linear solution if and only if the characteristic of the finite field does not belong to the given set.

Proof: The network in Fig. IV constructed for $q = p_1^{r_1} \cdot p_2^{r_2} \dots p_l^{r_l}$, where $r_1, r_2, \dots, r_l \in \mathbb{Z}^+$ is such a network. ■

V. CONCLUSION

The Fano and non-Fano networks have been used in the literature to show the insufficiency of linear network coding. In this paper, we have first constructed a network, named as the generalized Fano network, which for any set of primes $\{p_1, p_2, \dots, p_l\}$, has a vector linear solution if and only if the characteristic of the finite field belongs to $\{p_1, p_2, \dots, p_l\}$. This network reduces to the known Fano network as a special case. We have then constructed a network which for any set of primes $\{p_1, p_2, \dots, p_l\}$, has a vector linear solution if and only if the characteristic of the finite field does not belong to $\{p_1, p_2, \dots, p_l\}$. This network reduces to the non-Fano network as a special case.

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